Learning the Language of Mathematics

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> Just as everybody must strive to learn language and writing before he can use them freely for expression of his thoughts, here too there is only one way to escape the weight of formulas. It is to acquire such power over the tool that, unhampered by formal technique, one can turn to the true problems.
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> — Hermann Weyl [4]

This paper is about the use of language as a tool for teaching mathematical concepts. In it, I want to show how making the syntactical and rhetorical structure of mathematical language clear and explicit to students can increase their understanding of fundamental mathematical concepts. I confess that my original motivation was partly self-defense: I wanted to reduce the number of vague, indefinite explanations on homework and tests, thereby making them easier to grade. But I have since found that language can be a major pedagogical tool. Once students understand HOW things are said, they can better understand WHAT is being said, and only then do they have a chance to know WHY it is said. Regrettably, many people see mathematics only as a collection of arcane rules for manipulating bizarre symbols — something far removed from speech and writing. Probably this results from the fact that most elementary mathematics courses — arithmetic in elementary school, algebra and trigonometry in high school, and calculus in college — are procedural courses focusing on techniques for working with numbers, symbols, and equations. Although this formal technique is important, formulae are not ends in themselves but derive their real importance only as vehicles for expression of deeper mathematical thoughts. More advanced courses — such as geometry, discrete mathematics, and abstract algebra — are concerned not just with manipulating symbols and solving equations but with understanding the interrelationships among a whole host of sophisticated concepts. The patterns and relationships among these concepts

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constitute the "true problems" of mathematics. Just as procedural mathematics courses tend to focus on "plug and chug" with an emphasis on symbolic manipulation, so conceptual mathematics courses focus on proof and argument with an emphasis on correct, clear, and concise expression of ideas. This is a difficult but crucial leap for students to make in transitioning from rudimentary to advanced mathematical thinking. At this stage, the classical trivium of grammar, logic, and rhetoric becomes an essential ally.

There is, in fact, a nearly universally accepted logical and rhetorical structure to mathematical exposition. For over two millennia serious mathematics has been presented following a format of definition-theorem-proof. Euclid's *Elements* from circa 300 BC codified this mode of presentation which, with minor variations in style, is still used today in journal articles and advanced texts. There is a definite rhetorical structure to each of these three main elements: definitions, theorems, and proofs. For the most part, this structure can be traced back to the Greeks, who in their writing explicitly described these structures. Unfortunately, this structure is often taught today by a kind of osmosis. Fragmented examples are presented in lectures and elementary texts. Over a number of years, talented students may finally unconsciously piece it all together and go on to graduate school. But the majority of students give up in despair and conclude that mathematics is just mystical gibberish

With the initial support of a grant from Clemson's Pearce Center for Technical Communication and the long-term moral support of the Communication Across the Curriculum program, I have been working for several years now on developing teaching strategies and developing teaching materials for making the syntactical and logical structure of mathematical writing clear and explicit to students new to advanced mathematics. The results have been gratifying: if the rules of the game are made explicit, students can and will learn them and use them as tools to understand abstract mathematical concepts. Several years ago, I had the opportunity of sharing these ideas with the Occasional Seminar on Mathematics Education at Cornell, and now through this paper, I hope to share them with a wider audience.

> *One should NOT aim at being possible to understand, but at being IMPOSSIBLE to misunderstand.*
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> — Quintilian, circa 100 AD

The use of language in mathematics differs from the language of ordinary speech in three important ways. First it is nontemporal — there is no past, present, or future in mathematics. Everything just "is". This presents difficulties in forming convincing examples of, say, logical principles using ordinary subjects, but it is not a major difficulty for the student. Also, mathematical language is devoid of emotional content, although informally mathematicians tend to enliven their speech with phrases like "Look at the subspace killed by this operator" or "We want to increase the number of good edges in the coloring." Again, the absence of emotion from formal mathematical discourse or its introduction in informal discourse presents no difficulty for students.

The third feature that distinguishes mathematical from ordinary language, one which causes enormous difficulties for students, is its precision. Ordinary speech is full of ambiguities, innuendoes, hidden agendas, and unspoken cultural assumptions. Paradoxically, the very clarity and lack of ambiguity in mathematics is actually a stumbling block for the neophyte. Being conditioned to resolving ambiguities in ordinary speech, many students are constantly searching for the hidden assumptions in mathematical assertions. But there are none, so inevitably they end up *changing* the stated meaning — and creating a misunderstanding. Conversely, since ordinary speech tolerates so much ambiguity, most students have little practice in forming clear, precise sentences and often lack the patience to do so. Like Benjamin Franklin they seem to feel that mathematicians spend too much time "distinguishing upon trifles to the disruption of all true conversation."

But this is the price that must be paid to enter a new discourse community. Ambiguities can be tolerated only when there is a shared base of experiences and assumptions. There are two options: to leave the students in the dark, or to tell them the rules of the game. The latter involves providing the experiences and explaining the assumptions upon which the mathematical community bases its discourse. It requires painstaking study of details that, once grasped, pass naturally into the routine, just as a foreign language student must give meticulous attention to declensions and conjugations so that he can use them later without consciously thinking of them. The learning tools are the same as those in a language class: writing, speaking, listening, memorizing models, and learning the history and culture. Just as one cannot read literature without understanding the language, similarly in mathematics (where "translation" is not possible) this exacting preparation is needed before one can turn to the true problems. Thus it has become an important part of all my introductory courses, both at the undergraduate and graduate level.

This paper is a report on my efforts to make the rhetorical and syntactical structure of mathematical discourse explicit and apparent to the ordinary student. For concreteness sake, it is based on examples from a College Geometry course for juniors majoring in Secondary Mathematics Education. The same principles and goals apply, however, from freshman discrete mathematics for computer science majors to the linear algebra course for beginning math graduate students. As such it is about teaching and learning the tool of language in mathematics and not about grappling with the deeper problems such as the discovery of new mathematics or the heuristic exposition of complex mathematical ideas or the emotional experience of doing mathematics. As important as these deeper problems are, they cannot be approached without first having power over the tool of language. Mastering the trivium is necessary before the quadrivium can be approached.

> *Mathematics cannot be learned without being understood — it is not a matter of formulae being committed to memory but of acquiring a capacity for systematic thought.*
> — Peter Hilton [3]

Systematic thought does not mean reducing everything to symbols and equations — even when that is possible. Systematic thought also requires precise verbal expression. Since serious mathematics is usually communicated in the definition-theorem-proof format, the first step in learning the formal communication of mathematics is in learning definitions. For this reason, and because it requires the least technical sophistication, I will illustrate my general methodology with definitions. Although the examples below are kept elementary for the sake of the general reader, the principles they illustrate become even more critical the more advanced the material. This is sometimes a difficult point for students, who may not understand the need for meticulous precision with elementary concepts. But to have the technique needed to deal with complicated definitions, say the definitions of equivalence relations or of continuity, it is necessary to first practice with simple examples like the definition of a square.

Let us begin with a definition of definitions and some examples of good and bad definitions. A definition is a *concise* statement of the *basic* properties of an object or concept which *unambiguously identify* that object or concept. The italicized words give the essential characteristics of a good definition. It should be *concise* and not ramble on with extraneous or unnecessary information. It should involve *basic* properties, ideally those that are simply stated and have immediate intuitive appeal. It should not involve properties that require extensive derivation or are hard to work with. In order to be *complete*, a definition must describe exactly the thing being defined — nothing more, and nothing less.

> GOOD DEFINITION: A rectangle is a *quadrilateral* all four of whose angles are right angles.

> POOR DEFINITION: A rectangle is a *parallelogram* in which the diagonals have the same length and all the angles are right angles. It can be inscribed in a circle and its area is given by the product of two adjacent sides.

This is not CONCISE. It contains too much information, all of which is correct but most of which is unnecessary.

> POOR DEFINITION: A rectangle is a *parallelogram* whose diagonals have equal lengths.

This statement is true and concise, but the defining property is not BASIC. This would work better as a theorem to be proved than as a definition. In mathematics, assertions of this kind are regarded as *characterizations* rather than as definitions.

> BAD DEFINITION: A rectangle is a *quadrilateral* with right angles.

This is AMBIGUOUS. With some right angles? With all right angles? There are lots of quadrilaterals that have some right angles but are not rectangles.

> UNACCEPTABLE DEFINITION:
> rectangle: has right angles

This is unacceptable because mathematics is written as English is written — in complete, grammatical sentences. Such abbreviations frequently hide major misunderstandings as will be pointed out below.

In Aristotle's theory of definition, every "concept is defined as a subclass of a more general concept. This general concept is called the *genus proximum*. Each special subclass of the *genus proximum* is characterized by special features called the *differentiae specificae*." [1, p. 135] We will refer to these simply as the *genus* and *species*. In each example above, the italicized word is the *genus*. In the case of rectangle, the genus is the class of quadrilaterals and the species is the requirement that all angles be right angles. One of the greatest difficulties students experience with new concepts is that they fail to understand exactly what the genus is to which the concept applies. The unacceptable definition above skirts this issue by avoiding the genus altogether. To illustrate the importance of genus, note that we cannot say:

These two points are parallel.
This triangle is parallel.
The function $f(x) = 3x + 1$ is parallel.
35 is a parallel number.

The term "parallel" has as its genus the class of pairs of lines (or more generally, pairs of curves). Any attempt to apply the word "parallel" to other kinds of objects, like pairs of points, triangles, functions, or numbers, results not in a "wrong" statement but in nonsense. Note that the nonsense is not grammatical, but rhetorical. The four statements above are all perfectly grammatical English sentences, but none of them makes sense because of the inappropriate genus. Students only rarely make nonsensical statements like the four above because the genus is on a sufficiently concrete level that confusion is unlikely. However, when several layers of abstraction are superimposed, as is common in modern mathematics, nonsense statements become more common. Let us look at a specific abstract example.

In geometry parallelism, congruence, and similarity are all examples of the general notion of an equivalence relation. Equivalence relations abstract the basic properties of "sameness" or equality — for example, similar triangles have the same shape and parallel lines have equal slopes. Euclid includes one such property of equivalence relations as the first of his common notions: "Things which are equal to the same thing are also equal to one other." [3] In modern terms, this property is called "transitivity" and is enunciated formally as follows:

> A relation R on a set X is transitive if and only if for all choices of three elements a, b, and c from X, **if** a is related to b and b is related to c, **then** a must also be related to c.

Let us look at this definition from the standpoints of rhetoric, grammar, and logic. Rhetorically, there are three layers of abstraction in this definition: first, the objects or elements (which are abstract rather than definite), then the set X of such objects, and finally the relation R on this set. Students struggling with these layers of abstraction tend to get them confused and may say:

"a, b, and c are not transitive but e, f, and g are."
"The set X is transitive."

Such statements do not make sense because they attempt to apply the term "transitive" at a lower layer of abstraction than its genus requires. Although it may be possible to guess what the student has in mind, it is important to stress that this is not enough, as the Quintilian quote emphasizes.

The definition of transitivity also illustrates the absence of ambiguity. There is no hidden assumption that a is related to b. There is no hidden assumption that a and c must be different. These assumptions are not left up to the discretion of the student or the whim of the professor. They are simply not there. Yet these assumptions are often tacitly made by students trying to understand transitivity.

Grammatically, students have a tendency to use the active voice "a relates to b" rather than the passive "a is related to b", which is standard mathematical usage. Attention to this single, simple linguistic detail seems to heighten the focus on listening for proper usage and as a consequence proper understanding. Students who are attentive and disciplined enough to pick up this minor detail, which incidentally I repeatedly stress, generally are more secure with the concepts and more likely to apply them correctly. Shallow listening leads to shallow understanding. Here the difference is not a significant one conceptually, but it is a difference which is universal in the culture of mathematical discourse and thus is a shibboleth for distinguishing a "native speaker" from an outsider.

Of course, understanding the definition of transitivity also requires understanding the logical structure of the species. In this case, the species involves two logical connectives: AND (logical conjunction) and IF ... THEN (implication) preceded by a universal quantifier FOR ALL. All of these present major difficulties for many students due to the comparative sloppiness of ordinary speech. For example, "any" is an ambiguous word since it can be used in both the universal and existential senses:

Can anyone work this problem?	(existential quantifier)
Anyone can do it!	(universal quantifier)

For this reason I urge students to avoid the use of "any' when trying to learn the use of quantifiers. Although much more could be said on these issues, for brevity let me turn immediately to the one which is by far most important and most difficult: implication.

Implications are the backbone of mathematical structure. Many definitions (like transitivity) involve implications and almost all theorems are implications with a hypothesis and a conclusion. Like the Eskimo "snow," the phenomenon is so pervasive in mathematical culture that we have evolved many different ways of expressing it. Here are eight different but equivalent ways of stating that squares are rectangles, with names for some of the variations given on the side:

1) If a figure is a square, then it is a rectangle.	Hypothetical
2) A figure is a square only if it is a rectangle.	
3) A figure is a rectangle whenever it is a square.	
4) All squares are rectangles.	Categorical
5) For a figure to be a square, it must necessarily be a rectangle.	Necessity
6) A sufficient condition for a figure to be a rectangle is that it be a square.	Sufficiency
7) A figure cannot be a square and fail to be a rectangle.	Conjunctive
8) A figure is either a rectangle or it is not a square.	Disjunctive

There are three major issues involved in understanding implications. Two of these are purely logical:

1) realizing that an implication is not the same as a conjunction:

"If quadrilateral ABCD is a square, then it is a rectangle."
is not the same as
"Quadrilateral ABCD is a square and a rectangle."

2) realizing that an implication is not the same as its converse:

"If quadrilateral ABCD is a square, then it is a rectangle."
is not the same as
"If quadrilateral ABCD is a rectangle, then it is a square."

The third issue is a more subtle rhetorical issue involving a grasp of the relationship between premise and conclusion. The relationship is not one of causality, and the premise and conclusion can be implicit in a turn of phrase that is not an explicit if-then statement. An excellent exercise is to give students a dozen or so implications, expressed in different ways, and ask them to find the premise and conclusion in each. Then ask them to reformulate each implication in several different ways, just as I did above for "Squares are rectangles." It is not necessary, and in fact in some ways undesirable, for the students to understand the meaning of the statements. The point here is that these are syntactical exercises, and it is enough to have a feel for the language and an understanding of syntax to be successful. It does not depend on the actual meaning. At this point as in the learning of definitions, I stress that the results must read and sound like good English sentences.

How is all of this implemented in the classroom? As I said above, I proceed similarly to teaching a foreign language. Early in the semester, I present the students with a list of roughly twenty common geometrical terms, such as, circle, square, trapezoid and midpoint, and for homework ask them to write out definitions. I provide them with the following "Guidelines for Definitions in Good Form":

1. A definition MUST be written as a complete, grammatically correct English sentence.
2. A definition MUST be an "if and only if" statement.
3. A definition MUST have a clearly stated ***genus*** and a clearly stated ***species***.
4. The quantifiers in a good definition MUST be explicitly and clearly stated.
5. The term being defined MUST be underlined.

The next few class periods are spent with students putting their definitions on the board. The class and I critique them according to the principles outlined above. This invariably brings to the fore many issues, ranging from a reluctance to write in complete sentences and a decided preference for symbols over words to the syntactical issues described above. Many misconceptions can be brought to light and usually corrected. I also call on students to state definitions verbally. By engaging both speaking and writing, I hope to more deeply and actively penetrate the students' thinking.

We also explore the meaning of the definitions, the range of choices available, and some of the history involved. For example, Aristotle (384 - 322 BC) insisted that the subclasses (species) of each genus be disjoint: they could not overlap and one subclass could not include another. Thus for Aristotle, a square was NOT a rectangle. [1, p. 136] From the modern point of view this is inconvenient. Virtually everything one wants to prove about non-square rectangles also holds for squares, so it is a nuisance to have to state and prove two separate theorems. The modern standard is that squares are special cases of rectangles, so theorems about rectangles also apply to squares.

Finally, students are assigned to groups, first to provide feedback on the members' definitions and later to compile as a group a list of "standard" definitions in good form for all the given terms.

I do not require students to memorize common geometric definitions, but when we reach the abstraction of transitivity and equivalence relations, I provide models which must be memorized. There are two main reasons for this. First, it is not possible to have a good class discussion involving these concepts if students must constantly flip through their notes to look up the definitions. Second, the definitions I provide are models of good mathematical expression, something which is often lacking in elementary texts. Students can use these models to help build their own definitions (and later, theorems and proofs), but most importantly, repeating them out loud and memorizing them helps develop an ear for how correct mathematical discourse should sound.

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗ ΕΙΣΙΤΩ

"Let no one ignorant of geometry enter here"
— Plato, now the Motto of the American Mathematical Society

In conclusion, I want to confess what my real goals are in teaching this material. In a society in which information is passed in 60 second sound bites and reasoning limited to monosyllabic simple sentences, careful, analytic thinking is in danger of extinction. And this is a grave danger in a democratic society beset by a host of very complex moral and social problems. When geometry passed from the pragmatic, monarchical Egyptian surveyors to the democratic Greek philosophers nearly three millennia ago, its purpose changed. True, geometry (and more generally mathematics) has been many practical applications. But that is not why geometry has retained a universal place in the curriculum. It has been taught to teach reasoning and intellectual discipline. This why Plato placed his famous motto over the academy door. That is why Abraham Lincoln studied Euclid. And that remains my main goal in teaching.

Notes

1. Lucas Bunt, P. S. Jones, and J. D. Bedient, *The Historical Roots of Elementary Mathematics*, Dover, New York, 1988.

2. Euclid, *The Thirteen Books of The Elements* (Sir Thomas L. Heath, trans.), Volume I, Dover, New York, 1956, p. 222..

3. Peter Hilton, "A Job on Our Hands" in *FOCUS*, Newsletter of the MAA, March, 1986.

4. Herman Weyl, *Space-Time-Matter*, New York, Dover, 1922.